SOME SEMIGROUPS ON AN n-CELL

BY

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The purpose of this paper is to prove a theorem which is a generalization of a theorem proved by the author in [5]. The latter theorem is a special case of the one presented here. The theorem to be proved is:

THEOREM. *Let S be a semigroup which is topologically a closed n-cell, $n \geqq 2$. Suppose for x and y in B, the bounding $(n-1)$-sphere of S, $xy=x$.*

Then: (1) *If $S=K$, the minimal ideal of S, then S consists entirely of left zeros, that is, $xS=x$ for each x in S.*

(2) *If $S\neq K$, then K is a deformation retract of S and K consists entirely of left zeros for S. Also there exists in S an I-semigroup T with the following properties*:

(i) *$S\backslash K^0=BT$, where K^0 denotes the interior of K.*

(ii) *If b_1 and b_2 are in B and t_1 and t_2 belong to T and if $b_1t_1=b_2t_2$ then $t_1=t_2$.*

(iii) *For b_1 and b_2 in B, t_1 and t_2 in T, $(b_1t_1)(b_2t_2)=b_1(t_1t_2)$.*

For definitions and background material the reader is referred to [6; 11.]

The proof of the theorem is divided into a sequence of lemmas throughout which the hypotheses of the theorem are assumed to hold. The case $S=K$ is easily disposed of in Lemmas 1, 2 and 3. The remainder of the lemmas is devoted to the case $S\neq K$. In this case, the general idea is to prove that the relation, $\leqq$, on Q the Rees quotient of S by the ideal K, defined by $a\leqq b$ if and only if $a=bc$ for some c in Q is a partial order on Q. Knowing this relation is a partial order, it is possible to construct an I-semigroup J in Q so that $Q=\pi(B)J$ where π is the natural map from S onto Q. This I-semigroup J is then "lifted" into S and it is shown that the I-semigroup T where $\pi(T)=J$ satisfies the conclusion of the theorem.

LEMMA 1. *Each element of B is a right identity for S. If $s\in S$ and n a positive integer then there exists an element $a\in S$ such that $a^n=s$.*

Proof. The proof of this lemma depends on the following theorem [4]: *If α is a continuous function from S to S such that α is the identity on B, then α maps S onto S.*

To prove the first part of the lemma, let $b_0\in B$ be a fixed element of B

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and define $\alpha: S \to S$ by $\alpha(x) = xb_0$. Then for b in B, by hypothesis, $\alpha(b) = bb_0 = b$, hence by the above theorem, α maps S onto S. Thus $Sb_0 = S$ and since b_0 is an idempotent it follows immediately that b_0 is a right identity for all of S. Since b_0 was arbitrary in B, the first part of the lemma follows.

For the remainder of the lemma let n be a fixed positive integer and define $\alpha: S \to S$ by $\alpha(x) = x^n$ for $x \in S$. Since B consists of idempotents α is the identity on B and hence maps S onto S. This, however, implies that each element of S has an nth root in S which is the statement of the lemma.

LEMMA 2. *For x in S there exists an idempotent e in S such that $ex = x = xe$.*

Proof. Let p belong to S and let $\{p_n\}$ be a sequence of elements in S defined in the following way: $p_0 = p$, and $(p_n)^2 = p_{n-1}$. Such a sequence exists by Lemma 1. Let $Z(\{p_n\})$ be defined as in [5] and let e be the idempotent in $Z(\{p_n\})$. The author proves in [5] that e acts as a two-sided identity for all of $\{p_n\}$ and, in particular $ep = p = pe$ which is as required by the lemma.

LEMMA 3. *If $S = K$, then $xS = x$ for each x in S.*

Proof. Since S is topologically a closed n-cell, each proper retract of S has fixed-point property. By Wallace [9] therefore S is a group or $K \subset E$. Clearly S is not a group, so $S = K$ consists entirely of idempotents. Also by Wallace [9], $eSe = e$ for each $e \in E$, thus for $b \in B$, it follows that $b = bSb = bS$. Now for arbitrary x in S by Lemma 1, $xb = x$ for $b \in B$, hence $xS = (xb)S = x(bS) = xb = x$ and the lemma is established.

In the remainder it will be assumed that $S \neq K$.

LEMMA 4. *$S \backslash K$ is connected.*

Proof. Wallace proved in [8] that $H^p(S) \approx H^p(K)$ and since S is a closed n-cell we have $H^p(K) = 0$ for all $p > 0$. In particular $H^{n-1}(K) = 0$, hence K does not cut R^n [4] and since K is contained in the interior of S, K does not cut S.

DEFINITION. For x and y in S with $x \notin By$ define $n(By, x)$, the index of By relative to x, as defined by Mostert and Shields in [6]. That is:

When $x \notin By$, the mapping $f: B \to S \backslash x$ defined by $f(b) = by$ induces a homomorphism $f^*: H^{n-1}(S \backslash x) \to H^{n-1}(B)$ where $H^{n-1}(A)$ denotes the $(n-1)$-Čech cohomology group of A with integer coefficients. Since $H^{n-1}(B)$ is isomorphic to the integers there exists a least positive integer k such that k generates $f^*(H^{n-1}(S \backslash x))$. For such a pair x and y in S define $n(By, x)$ to be k.

LEMMA 5. *If A is a connected space and $\sigma: A \to S$ and $\tau: A \to S$ are continuous functions such that $\tau(t') \notin B\sigma(t)$ for each t and t' in A, if $\sigma(A)$ is compact or if τ is a constant, then $n(B\sigma(t), \tau(t)) = n(B\sigma(t'), \tau(t'))$ for t and t' in A.*

Proof. Assume $\sigma(A)$ is compact. Since A is connected it suffices to show that for each t in A there exists an open set U containing t such that for x

and y in U, $n(B\sigma(x), \tau(x)) = n(B\sigma(y), \tau(y))$. To show the existence of such U, let t_0 belong to A. By hypothesis $\tau(t_0)$ is not an element of $B\sigma(A)$ so there exists an open n-cell O_1 in S such that $\tau(t_0) \in O_1$ and $O_1^* \cap B\sigma(A) = \square$. Hence $B\sigma(A) \subset S \backslash O_1^*$. By hypothesis τ is a continuous function so there exists an open set U in A containing t_0 with $\tau(U) \subset O_1$. The claim is now made that $n(B\sigma(t_0), \tau(t_0)) = n(B\sigma(s), \tau(s))$ for each s in U. To establish the claim let s belong to U and define maps λ_s, λ_{t_0}, m_0, I and J in the following way:

$$\lambda_s: B \to B \times A \quad \text{by} \quad \lambda_s(b) = (b, s),$$
$$\lambda_{t_0}: B \to B \times A \quad \text{by} \quad \lambda_{t_0}(b) = (b, t_0),$$
$$m_0: B \times A \to S \quad \text{by} \quad m_0(b, t) = b\sigma(t),$$

and I and J are the injection maps from $S \backslash O_1^*$ to $S \backslash \tau(s)$ and $S \backslash \tau(t_0)$ respectively. Then it is easily seen that the mappings

$$\theta_s: B \to S \backslash \tau(s) \text{ defined by } \theta_s(b) = b\sigma(s)$$

and

$$\theta_{t_0}: B \to S \backslash \tau(t_0) \text{ defined by } \theta_{t_0}(b) = b\sigma(t_0)$$

are given by

$$\theta_s = Im_1 \lambda_s \quad \text{and} \quad \theta_{t_0} = Jm_0 \lambda_{t_0}$$

where m_1 is m_0 with the range restricted to $S \backslash O_1^*$.

The following sequences now arise from these functions:

$$H^{n-1}(S \backslash \tau(s)) \xrightarrow{I^*} H^{n-1}(S \backslash O_1^*) \xrightarrow{\lambda_s^* m_1^*} H^{n-1}(B)$$

and the same sequence obtained by replacing s by t_0 and I^* by J^*.

Since O_1 is an open n-cell for any y in O_1 the injection map from $S \backslash O_1^*$ into $S \backslash y$ induces an isomorphism from $H^{n-1}(S \backslash y)$ onto $H^{n-1}(S \backslash O_1^*)$ [1]. Hence I^* and J^* are isomorphisms onto. By S. T. Hu [3], $\lambda_s^* = \lambda_{t_0}$ so it follows that

$$\lambda_s^* m_1^* = \lambda_{t_0}^* m_1^*.$$

Looking at the above sequences it is easily seen that

$$\theta_s^*(H^{n-1}(S \backslash \tau(s))) = \theta_{t_0}^*(H^{n-1}(S \backslash \tau(t_0))).$$

Since I^* and J^* are isomorphisms onto and

$$\theta_s^* = \lambda_s^* m_1^* I^*, \qquad \theta_{t_0}^* = \lambda_{t_0}^* m_1^* J^*.$$

From this we obtain that

$$n(B\sigma(t_0), \tau(t_0)) = n(B\sigma(s), \tau(s))$$

and the first part of the proof of the lemma is complete. The remainder of the proof follows similarly.

LEMMA 6. *If x belongs to $S\backslash B$, then $n(Bb, x)=1$ for each $b\in B$.*

Proof. Let b_0 belong to B and let $x\in S\backslash B$. Define θ from B to $S\backslash x$ by $\theta(b)=bb_0$. By hypothesis on the multiplication in B, $\theta(b)=b$ for each b in B. Let $\delta: S\backslash x\rightarrow B$ be a continuous function from $S\backslash x$ onto B such that $\delta(b)=b$ for each b in B. If ϕ denotes the function from B onto B defined by $\phi(b)=\delta\theta(b)$ then ϕ is the identity function so that

$$\phi^*: H^{n-1}(B) \rightarrow H^{n-1}(B)$$

is an isomorphism. From this it follows that

$$\theta^*: H^{n-1}(S\backslash x) \rightarrow H^{n-1}(B)$$

is onto since

$$\phi^* = \theta^*\delta^*.$$

Thus by the definition of $n(Bb_0, x)$ we have $n(Bb_0, x)=1$ and the lemma is established.

LEMMA 7. *For b in B and x in S with $b\notin Bx$, $n(Bx, b)=0$.*

Proof. Let $\theta: B\rightarrow S\backslash b$ be defined by $\theta(s)=sx$. Since $b\notin Bx$ it follows that $Bx\subset S\backslash B$. For if $Bx\cap B$ were nonvoid, then for $y\in Bx\cap B$ there would exist $b_0\in B$ such that $y=b_0x$ and in virtue of the multiplication in B, that $b=by=b(b_0x)=(bb_0)x=bx$ contrary to the assumption that $b\notin Bx$. Hence Bx is a closed subset of S contained in $S\backslash B$. Since B is the boundary of S relative to R^n there exists a subset S_0 of S with the following properties: S_0 is closed, S_0 is topologically equivalent to S and $Bx\subset S_0\subset S\backslash B$. Now define functions i_1 and i_2 by

$$i_1: Bx \rightarrow S_0 \quad \text{and} \quad i_1(y) = y \qquad \text{for } y \in Bx,$$

$$i_2: S_0 \rightarrow S\backslash b \quad \text{and} \quad i_2(y) = y \qquad \text{for } y \in S_0.$$

Also define

$$\theta_1: B \rightarrow Bx \quad \text{by} \quad \theta_1(y) = yx \qquad \text{for } y \in B.$$

Clearly $\theta=i_2i_1\theta_1$ so that $\theta^*=\theta_1^*i_1^*i_2^*$. Looking at the sequence defined by these functions it follows that θ^* is the zero homomorphism, for we have:

$$H^{n-1}(S\backslash b) \xrightarrow{i_2^*} H^{n-1}(S_0) \xrightarrow{i_1^*} H^{n-1}(Bx) \xrightarrow{\theta_1^*} H^{n-1}(B)$$

and $H^{n-1}(S_0)=0$. From this it follows that $n(Bx, b)=0$.

LEMMA 8. *For $a\in S\backslash K$, a belongs to BS. Thus each element of $S\backslash K$ has a two-sided identity belonging to B.*

Proof. Suppose there exists an element a_0 in $S\backslash K$ such that $a_0\notin BS$. Let

$k \in K$ and $f \in B$ be fixed. Clearly $Bk \cap S \backslash K = \Box$ and since $S \backslash K$ is connected it follows from Lemma 5, taking $A = S \backslash K$, $\tau =$ identity and $\sigma =$ constant map k, that $n(Bk, x) = n(Bk, f)$ for each $x \in S \backslash K$. But a_0 belongs to $S \backslash K$ so that $n(Bk, f) = n(Bk, a_0) = 0$ by Lemma 7.

Now using the assumption that $a_0 \notin BS$, it follows in a similar way from Lemma 5, taking $A = S$, $\sigma =$ identity, and $\tau =$ constant map a_0, that $n(Bf, a_0) = n(Bk, a_0)$. Hence by Lemma 6, $n(Bk, a_0) = 1$. This contradiction establishes the fact that $a_0 \in BS$. The remainder of the lemma follows quite easily since each element of B is an idempotent and a right identity for all of S.

LEMMA 9. *If $a \in S \backslash K$, then $Ba \neq a$.*

Proof. To prove this lemma let us assume that $Ba = a$ for some element a in $S \backslash K$. The claim is now made that with this assumption $B(S \backslash K) = S$. If this were not the case then there would exist an element p in S with $B(S \backslash K) \subset S \backslash p$. Since $B \subset B(S \backslash K)$ it follows that $p \notin B$ hence it is possible to define a function $\delta: S \backslash p \to B$ such that δ is continuous and $\delta(b) = b$ for each b in B. Now for each x in $S \backslash K$ define a function $\theta_x: B \to B$ by $\theta_x(b) = \delta(bx)$. For each b in B, θ_b is the identity and for a, θ_a is a constant. From this it can be concluded that the identity function on B is null-homotopic, since $S \backslash K$ is connected. This contradiction establishes the fact that $B(S \backslash K) = S$.

Since $B(S \backslash K) = S$ and K is nonempty, there exists an element g in B and x in $S \backslash K$ such that $gx \in K$. By Lemma 8, there exists an element b in B with $bx = x$. Hence $x = bx = (bg)x = b(gx) \in BK \subset K$ contrary to the fact that $x \in S \backslash K$. From this we obtain that $Ba \neq a$ for each a in $S \backslash K$.

LEMMA 10. *For a in $S \backslash K$, $J_a = Ba$ where J_a denotes the set of elements in S generating the same two-sided ideal as a.*

Proof. Before proving this lemma let us note that the ideal generated by an element x in $S \backslash K$ is SxS. If $J(x)$ denotes the ideal generated by x then $J(x) = x \cup xS \cup Sx \cup SxS = SxS$ since x has a two-sided identity in S.

It follows from Lemma 1 that $Ba \subset J_a$ for if $b \in B$ then $J(ba) = S(ba)S = (Sb)aS = SaS = J(a)$ so that $ba \in J_a$.

It remains only to show that $J_a \subset Ba$. First let us note that $Ba \cap K = \Box$ since $a \notin K$, as in the proof of Lemma 9. Hence $K \subset S \backslash Ba$, and if P denotes the component of $S \backslash Ba$ containing K it follows from Wallace [9] that $P^* \backslash P = Ba$. For an element p in $P \backslash K$, $Bp \cap Ba = \Box$ for if not then $b_1 p = b_2 a$ for elements b_1 and b_2 in B. By Lemma 8 there exists b in B such that $bp = p$, hence $p = bp = (bb_1)p = b(b_1 p) = b(b_2 a) = (bb_2)a = ba$ contrary to the fact that $p \in P$. Hence Bp does not meet Ba and since $Bp \cap P$ contains p, Bp is connected and P is a component of $S \backslash Ba$ we have $Bp \subset P$. By assumption $p \notin K$, hence $K \subset S \backslash Bp$, as in the proof of Lemma 9. Let Q be the component of $S \backslash Bp$ containing K. Clearly $K \subset Q \subset P$ and as before $Q^* \backslash Q = Bp \subset P$. Let $I(p) = J(p) \backslash J_p$. Then $I(p)$ must contain K, $I(p)$ does not meet Bp and by

Wallace [9], $I(p)$ is connected and $I(p)^*=J(p)$. The last statement follows from the fact that $Bp\subset J_p$ and by Lemma 9, $Bp\neq p$ so that $J_p\neq p$. Since $I(p)$ is connected and contains K, $I(p)\subset Q$, hence $J(p)=I(p)^*\subset Q^*=Q\cup Bp \subset P$. From this discussion we obtain that $J(p)\subset P$ for each $p\in P\backslash K$, hence $J_a\cap P=\square$. But $I(a)\subset P$ so that $J(a)=I(a)^*\subset P^*=P\cup Ba$, therefore $J_a\subset Ba$ and Lemma 10 is established.

Definition. For a and b in $S\backslash K$ define $a\leqq b$ if and only if there exists an element c in $S\backslash K$ such that $a=bc$.

Lemma 11. *$\leqq$ as defined above is a partial order on $S\backslash K$.*

Proof. (i) Since $a\in S$, $a=af$ for $f\in B$, so that $a\leqq a$ and $\leqq$ is reflexive.

(ii) If a and b belong to $S\backslash K$ and $a\leqq b$, and $b\leqq a$, then there exist elements c and d in $S\backslash K$ such that $a=bc$ and $b=ad$. Thus $aS=(bc)S=b(cS) \subset bS=(ad)S=a(dS)\subset aS$, or $aS=bS$. Hence $SaS=SbS$ so that $J_a=J_b$ and by Lemma 10, $Ba=Bb$. Since a and b both belong to $S\backslash K$ there exist elements e and f in B such that $ea=a$ and $fb=b$. Now $a\in Ba=Bb$ so that $a=gb$ for some $g\in B$. From these equalities it follows that $a=ea=e(gb)=(eg)b=eb =e(ad)=(ea)d=ad=b$ so that $\leqq$ is antisymmetric.

(iii) Clearly $\leqq$ is transitive.

(i), (ii) and (iii) show that $\leqq$ is a partial order on $S\backslash K$.

Notation. For the minimal ideal K in S, let Q denote the Rees Quotient of S by K and let π denote the natural map from S to Q. By Rees [7], Q is a compact connected semigroup with zero, $\pi(K)$, and π is continuous and a homomorphism.

It should be noted at this point that π restricted to $S\backslash K$ is an iseomorphism. For this reason, in the discussion that follows $S\backslash K$ and $\pi(S\backslash K)$, the former a subset of S and the latter a subset of Q will be considered the same. This identification will make the discussion simpler and somewhat shorter.

Lemma 12. *There exists an I-semigroup $J\subset Q$ such that $Q=BJ$.*

Proof. Let f be a fixed element in B. Then fQ is a compact connected semigroup with identity f and zero $\pi(K)$. Define a partial order on fQ by $a\leqq b$ if and only if $a=bc$ for some $c\in fQ$. By Lemma 11, the fact that f is a right identity for all of S and the fact that $\pi(K)$ is a zero for fQ, it is easily seen that $\leqq$ is a closed partial order on fQ. Hence by Koch [2] there exists an I-semigroup $J\subset fQ$ with endpoints f and $\pi(K)$.

The next step in the proof is to show that $BJ=Q$. If it were the case that $S=BJ_0\cup K$, where $J_0=J\backslash\pi(K)$, it would follow immediately that $Q=\pi(S) =\pi(BJ_0)\cup\pi(K)=BJ$. Hence it suffices to show that $S=BJ_0\cup K$.

Let us assume, to the contrary, that there exists an element p in S with p not in $BJ_0\cup K$. Since J_0 is a half-open interval and $J=J_0\cup\pi(K)$ is closed there exists an element k_0 in K with $J_0\subset J_0\cup k_0\subset J_0^*$, where J_0^* denotes the closure of J_0 in S. Since J_0 is connected, $J_0\cup k_0$ is connected and by assumption $p \notin B(J_0 \cup k_0)$. Thus by Lemmas 5 and 7, $n(Bp, k_0) = n(Bp, f) = 0$. Now $p \in S \backslash K$ and since $S \backslash K$ is connected and $(B(S \backslash K)) \cap K = \square$, it follows that $n(Bp, k_0) = n(Bf, k_0) = 1$, again by Lemmas 5 and 6. This is a contradiction so p must belong to $B(J_0 \cup k_0)$. With the preceding remarks the lemma is established.

LEMMA 13. *There exists an element k_0 in $K \backslash K^0$ such that if T denotes J_0^*, then $T = J_0 \cup k_0$ and $K \backslash K^0 = Bk_0$.*

Proof. From the definition of J_0 we see that $\pi(J_0^* \backslash J_0) = \pi(K)$, hence $J_0^* \backslash J_0 \subset K$. Now let $k_0 \in J_0^* \backslash J_0$. The claim is made that $K \backslash K^0 = Bk_0$. To prove this claim let $k = gk_0$ for some $g \in B$ and assume $k \in U$, an open set. Since $k = gk_0$ and $k \in U$, there must exist open sets V_0 and V_1 containing g and k_0, respectively, such that $V_0 V_1 \subset U$. Now $k_0 \in J_0^* \backslash J_0$ and V_1 is open containing k_0, hence there exists an element t in J_0 with $t \in V_1$. Since $t \in S \backslash K$, it follows that gt also belongs to $S \backslash K$ so that $U \cap S \backslash K \neq \square$. Since k was an arbitrary element in Bk_0, it follows that $Bk_0 \subset K \backslash K^0$.

Conversely, let $k \in K \backslash Bk_0$. If it can be shown that $k \in K^0$ then it will be established that $Bk_0 = K \backslash K^0$. To prove $k \in K^0$, let P be the component of $S \backslash Bk_0$ containing k. As before, since $J_0 \cup k_0$ is connected $n(Bk_0, k) = n(Bf, k) = 1$. If it were the case that $B \subset P$, then it would be true that $n(Bk_0, k) = n(Bk_0, f) = 0$ since P is connected and does not meet Bk_0. This is a contradiction to the above statement that $n(Bk_0, k) = 1$, hence B does not meet the component P. Thus the boundary of P relative to R^n is contained in Bk_0 which is a subset of K. Now if P is not contained in K, then K is a closed proper subset of $P \cup K$ containing the boundary of $P \cup K$. Hence

$$i^*: H^{n-1}(P \cup K) \rightarrow H^{n-1}(K)$$

is not onto where i^* is induced by the injection map [4]

$$i: K \rightarrow P \cup K.$$

By Wallace [8], however, $H^{n-1}(K) \approx H^{n-1}(S) = 0$, so that i^* is onto. Thus $P \cup K = K$, that is $P \subset K$. Since P is a component of an open set in S, P is also open and therefore $k \in P \subset K^0$. This completes the proof of the statement that $Bk_0 = K \backslash K^0$.

In order to complete the proof of this lemma it remains only to show that $T = J_0 \cup k_0$. By definition of T we have $T \subset fS$ since $J_0 \subset fS$ and therefore $T = J_0^* \subset (fS)^* = fS$. This shows that f is a two-sided identity for T. In the above argument it was shown that $J_0^* \backslash J_0 \subset K \backslash K^0 = Bk_0$. Now let $k \in T \backslash J_0$, then $k = gk_0$ for some $g \in B$ and $fk = k$, $fk_0 = k_0$. Hence $k = fk = f(gk_0) = (fg)k_0 = fk_0 = k_0$ so that $T \backslash J_0 = k_0$. Thus $T = J_0 \cup k_0$ and the proof of the lemma is complete.

LEMMA 14. *T is an I-semigroup with zero k_0 and identity f. Also $BT = S \backslash K^0$.*

Proof. Clearly T is a semigroup and an arc with zero k_0 and identity f. Also $S\backslash K^0 = S\backslash K \cup K\backslash K^0 = BJ_0 \cup Bk_0 = B(J_0 \cup k_0) = BT$. This concludes Lemma 14.

LEMMA 15. *For k in K, $kS = k$.*

Proof. First let us note that by Wallace [9], $K \subset E$ and $kSk = k$ for each $k \in K$. If $K^0 = \square$, then $Bk_0 = K$ so that $k_0K = k_0(Bk_0) \subset k_0Sk_0 = k_0$. Thus $k_0S = k_0$ since $k_0S \subset k_0K$. If $K^0 \neq \square$ then Bk_0, since it is the boundary of K relative to R^n is an $((n-1), G)$-rim for K, (see [10]). Hence by the dual of Wallace's theorem [10], if $k \in K$ and $(Bk_0)k = Bk_0$ it follows that $Kk = k$. Since $k_0^2 = k_0$, we have $(Bk_0)k_0 = Bk_0^2 = Bk_0$ so that $Kk_0 = K$. Hence $k_0S \subset k_0K = k_0(Kk_0) = k_0$.

In either case, $K^0 = \square$ or K^0 nonempty it has been shown that $k_0S = k_0$. Now let k be an arbitrary element of K. Then $k_0k = k_0$ so that $kk_0 = k(k_0k) = k$ since $kSk = k$. Hence $kK = (kk_0)K = k(k_0K) = kk_0 = k$ and it follows that $kS = k$ which concludes the proof of the lemma.

LEMMA 16. *Let t_0 and t_1 belong to T and let b_0 and b_1 be elements of B. Then $(b_0t_0)(b_1t_1) = b_0(t_0t_1)$ and if $b_0t_0 = b_1t_1$ then $t_0 = t_1$.*

Proof. This lemma follows immediately from the fact that f is an identity for T and $fb = f$ for each b in B.

LEMMA 17. *K is a deformation retract of S.*

Proof. Define $\theta: S \times T \to S$ by $\theta(s, t) = st$. T is a closed interval with endpoints f and k_0, $\theta(s, f) = sf = s$ and $\theta(s, k_0) = sk_0 \in K$. Also for $k \in K$, $\theta(k, k_0) = kk_0 = k$. Since θ is continuous it follows that K is a deformation retract of S.

With Lemma 17 the proof of the theorem is now complete.

EXAMPLE. An example of a semigroup described by the theorem and having a nontrivial kernel for $n = 2$ can be constructed as follows.

Let K_0 be a closed two-cell and B_0 the bounding 1-sphere of K_0. Define multiplication in K_0 by $xy = x$ for all x and y in K_0. Let T_0 be the closed unit interval with real multiplication. Then if $S = (K_0 \times \{0\}) \cup (B_0 \times T_0)$ and products are defined in S by coordinate-wise multiplication, S is a semigroup as described by the theorem, where B, of course, is $B_0 \times \{1\}$.

Clearly S is topologically a closed two-cell and is a semigroup with a nontrivial kernel $K = K_0 \times \{0\}$. If k_0 is a fixed element of B_0, then $T = \{k_0\} \times T_0$ is an I-semigroup which has the property that $S\backslash K^0 = BT$.

In this example, for $a \in S\backslash K$, the representation of $a = bt$ for $b \in B$ and $t \in T$ is unique. In [5], the author gives an example of such a semigroup described above but in it there exists an element in $S\backslash K$ for which this representation is not unique.

For $n = 2$, different examples may be constructed by varying the multiplication of the I-semigroup T_0. (See [6].)

For any integer $n>2$, examples can be constructed in a similar way. That is, let K_0 be a closed n-cell with B_0 the bounding $(n-1)$-sphere and follow the same construction as above.

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